Hausdorff Continuous Viscosity Solutions of Hamilton-Jacobi Equations

R Anguelov[1,2], S Markov[2,*], F Minani[3]

[1]Department of Mathematics and Applied Mathematics, University of Pretoria
[2]Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
[3]Department of Applied Mathematics, National University of Rwanda

Abstract. A new concept of viscosity solutions, namely, the Hausdorff continuous viscosity solution for the Hamilton-Jacobi equation is defined and investigated. It is shown that the main ideas within the classical theory of continuous viscosity solutions can be extended to the wider space of Hausdorff continuous functions while also generalizing some of the existing concepts of discontinuous solutions.

1   Introduction

The theory of viscosity solutions was developed for certain types of first and second order PDEs. It has been particularly useful in describing the solutions of PDEs associated with deterministic and stochastic optimal control problems [7], [3]. In its classical formulation, see [6], the theory deals with solutions which are continuous functions. The concept of continuous viscosity solutions was further generalized in various ways, e.g. see [3, Chapter V], [5], [4], to include discontinuous solutions with the definition of Ishii given in [8] playing a pivotal role. In this paper we propose a new approach to the treatment of discontinuous solutions, namely, by involving Hausdorff continuous (H-continuous) interval valued functions. In the sequel we will justify the advantages of the proposed approach by demonstrating that

- the main ideas within the classical theory of continuous viscosity solutions can be extended almost unchanged to the wider space of H-continuous functions
- the existing theory of discontinuous solutions is a particular case of that developed in this paper in terms of H-continuous functions
- the H-continuous viscosity solutions have a more clear interpretation than the existing concepts of discontinuous solutions, e.g. envelope viscosity solutions [3, Chapter V].

---

* Corresponding author: smarkov@bio.bas.bg

In order to simplify the exposition we will only consider first order Hamilton-Jacobi equations of the form

$$\Phi(x, u(x), Du(x)) = 0, \ x \in \Omega, \tag{1}$$

where Ω is an open subset of $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}$ is the unknown function, Du is the gradient of u and the given function $\Phi : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is jointly continuous in all its arguments.

The theory of viscosity solutions rests on two fundamental concepts, namely, of subsolution and of supersolution. These concepts are defined in various equivalent ways in the literature. The definition given below is formulated in terms of local maxima and minima. We will use the following notations

$$USC(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is upper semi-continuous on } \Omega\}$$

$$LSC(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is lower semi-continuous on } \Omega\}$$

Definition 1. *A function $u \in USC(\Omega)$ is called a viscosity subsolution of the equation (1) if for any $\varphi \in C^1(\Omega)$ we have $\Phi(x_0, u(x_0), D\varphi(x_0)) \leq 0$ at any local maximum point x_0 of $u - \varphi$. Similarly, $u \in LSC(\Omega)$ is called a viscosity supersolution of the equation (1) if for any $\varphi \in C^1(\Omega)$ we have $\Phi(x_0, u(x_0), D\varphi(x_0)) \geq 0$ at any local minimum point x_0 of $u - \varphi$.*

Without loss of generality we may assume in the above definition that $u(x_0) = \varphi(x_0)$ exposing in this way a very clear geometrical meaning of this definition: the gradient of the solution u of equation (1) is replaced by the gradient of any smooth function touching the graph of u from above, in the case of subsolution, and touching the graph of u from below, in the case of supersolution. This also establishes the significance of the requirement that a subsolution and a supersolution should respectively be upper semi-continuous and lower semi-continuous functions. More precisely, the upper semi-continuity of a subsolution u ensures that any local supremum of $u - \varphi$ is effectively reached at a certain point x_0, that is, it is a local maximum, with the geometrical meaning that the graph of u can be touched from above at $x = x_0$ by a vertical translate of the graph of φ. In a similar way, the lower semi-continuity of a supersolution u ensures that any local infimum of $u - \varphi$ is effectively reached at a certain point x_0 which means that the graph of u can be touched from below at $x = x_0$ by a vertical translate of the graph of φ.

Naturally, a solution should be required somehow to incorporate the properties of both a subsolution and a supersolution. In the classical viscosity solutions theory, see [6], a viscosity solution is a function u which is both a subsolution and a supersolution. Since $USC(\Omega) \bigcap LSC(\Omega) = C(\Omega)$, this clearly implies that the viscosity solutions defined in this way are all continuous functions.

The concept of viscosity solution for functions which are not necessarily continuous is introduced by using the upper and lower semi-continuous envelopes. For a locally bounded function u these are given by

$$S(u)(x) = \inf\{f(x) : f \in USC(\Omega), u \leq f\} = \inf_{\delta > 0} \sup\{u(y) : y \in B_\delta(x)\}, \tag{2}$$

$$I(u)(x) = \sup\{f(x) : f \in LSC(\Omega), u \geq f\} = \sup_{\delta > 0} \inf\{u(y) : y \in B_\delta(x)\}, \tag{3}$$

where $B_\delta(x)$ denotes the open δ-neighborhood of x in Ω. Using the fact that for any function $u : \Omega \to \mathbb{R}$ the functions $S(u)$ and $I(u)$ are always, respectively, upper semi-continuous and lower semi-continuous functions, a viscosity solution can be defined as follows, [8].

Definition 2. *A function $u : \Omega \to \mathbb{R}$ is called a viscosity solution of (1) if $S(u)$ is a viscosity subsolution of (1) and $I(u)$ is a viscosity supersolution of (1).*

The first important point to note about the advantages of the method in this paper is as follows. Interval valued functions appear naturally in the context of noncontinuous viscosity solutions. Namely, they appear as *graph completions*. Indeed, the above definition places requirements not on the function u itself but on its lower and upper semi-continuous envelopes or, in other words, on the interval valued function

$$F(u)(x) = [I(u)(x), S(u)(x)], \ x \in \Omega, \tag{4}$$

which is called the *graph completion* of u, see [12]. Clearly, Definition 2 treats functions which have the same upper and lower semi-continuous envelopes, that is, have the same graph completion, as identical functions. On the other hand, since different functions can have the same graph completion, a function can not in general be identified from its graph completion, that is, functions with the same graph completion are indistinguishable. Therefore, no generality will be lost if only interval valued functions representing graph completions are considered.

Let $\mathbb{A}(\Omega)$ be the set of all functions defined on an open set $\Omega \subset \mathbb{R}^n$ with values which are closed finite real intervals, that is,

$$\mathbb{A}(\Omega) = \{f : \Omega \to \mathbb{IR}\},$$

where $\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R}, \ \underline{a} \leq \overline{a}\}$. Identifying $a \in \mathbb{R}$ with the point interval $[a, a] \in \mathbb{IR}$, we consider $\mathbb{R}$ as a subset of $\mathbb{IR}$. Thus $\mathbb{A}(\Omega)$ contains the set $\mathcal{A}(\Omega) = \{f : \Omega \to \mathbb{R}\}$ of all real functions defined on Ω.

Let $u \in \mathbb{A}(\Omega)$. For every $x \in \Omega$ the value of u is an interval $[\underline{u}(x), \overline{u}(x)] \in \mathbb{IR}$. Hence, the function u can be written in the form $u = [\underline{u}, \overline{u}]$ where $\underline{u}, \overline{u} \in \mathcal{A}(\Omega)$ and $\underline{u}(x) \leq \overline{u}(x), \ x \in \Omega$. The function

$$w(f)(x) = \overline{u}(x) - \underline{u}(x), \ x \in \Omega,$$

is called width of u. Clearly, $u \in \mathcal{A}(\Omega)$ if and only if $w(f) = 0$. The definitions of the upper semi-continuous envelope, the lower semi-continuous envelope and the graph completion operator F given in (2), (3) and (4) for $u \in \mathcal{A}(\Omega)$ can be extended to functions $u = [\underline{u}, \overline{u}] \in \mathbb{A}(\Omega)$ in an obvious way, e.g. $S(u) = \inf_{\delta > 0} \sup\{z \in u(y) : y \in B_\delta(x)\}$. We recall here the concept of S-continuity associated with the graph completion operator, [12].

Definition 3. *A function $u = [\underline{u}, \overline{u}] \in \mathbb{A}(\Omega)$ is called S-continuous if $F(u) = u$, or, equivalently, $I(\underline{u}) = \underline{u}, \ S(\overline{u}) = \overline{u}$.*

Using the properties of the lower and upper semi-continuous envelopes one can easily see that the graph completions of locally bounded real functions on Ω comprise the set $\mathbb{F}(\Omega)$ of all S-continuous functions on Ω. Following the above discussion we define the concept of viscosity solution for the interval valued functions in $\mathbb{F}(\Omega)$.

Definition 4. *A function $u = [\underline{u}, \overline{u}] \in \mathbb{F}(\Omega)$ is called a viscosity solution of (1) if $\underline{u}$ is a supersolution of (1) and $\overline{u}$ is a subsolution of (1).*

A second advantage of the method in this paper is as follows. A function $u \in \mathcal{A}(\Omega)$ is a viscosity solution of (1) in the sense of Definition 2 if and only if the interval valued function $F(u)$ is a viscosity solution of (1) in the sense of Definition 4. In this way the level of the regularity of a solution u is manifested through the width of the interval valued function $F(u)$. It is well known that without any additional restrictions the concept of viscosity solution given in Definition 2 and by implication the concept given in Definition 4 is rather weak, [3]. This is demonstrated by the following example, which is also partially discussed in [3].

Example 1. Consider the equation

$$u'(x) = 1 \ , \quad x \in (0, 1). \tag{5}$$

The functions

$$v(x) = \begin{cases} x+1 & \text{if } x \in (0,1) \cap \mathbb{Q} \\ x & \text{if } x \in (0,1) \setminus \mathbb{Q} \end{cases} \qquad w(x) = \begin{cases} x & \text{if } x \in (0,1) \cap \mathbb{Q} \\ x+1 & \text{if } x \in (0,1) \setminus \mathbb{Q} \end{cases}$$

are both viscosity solutions of equation (5) in terms of Definition 2. The interval valued function $z = F(v) = F(w)$ given by $z(x) = [x, x+1]$, $x \in (0,1)$, is a solution in terms of Definition 4.

With the interval approach adopted here it becomes apparent that the distance between $I(u)$ and $S(u)$ is an essential measure of the regularity of any solution u, irrespective of whether it is given as a point valued function or as an interval valued function. If no restriction is placed on the distance between $I(u)$ and $S(u)$ we will have some quite meaningless solutions like the solutions in Example 1. On the other hand, a strong restriction like $I(u) = S(u)$ gives only solutions which are continuous. In this paper we consider solutions for which the Hausdorff distance, as defined in [12], between the functions $I(u)$ and $S(u)$ is zero, a condition defined through the concept of Hausdorff continuity.

2 The space of Hausdorff continuous functions

The concept of Hausdorff continuous interval valued functions was originally developed within the theory of Hausdorff approximations, [12]. It generalizes the concept of continuity of real function using a minimality condition with respect to inclusion of graphs.

Definition 5. *A function $f \in \mathbb{A}(\Omega)$ is called Hausdorff continuous, or H-continuous, if for every $g \in \mathbb{A}(\Omega)$ which satisfies the inclusion $g(x) \subseteq f(x)$, $x \in \Omega$, we have $F(g)(x) = f(x)$, $x \in \Omega$.*

As mentioned in the Introduction the concept of Hausdorff continuity is closely connected with the Hausdorff distance between functions. Let us recall that the Hausdorff distance $\rho(f, g)$ between two functions $f, g \in \mathbb{A}(\Omega)$ is defined as the Hausdorff distance between the graphs of the functions $F(f)$ and $F(g)$ considered as subsets of $\mathbb{R}^{n+1}$, [12]. Then we have

$$f = [\underline{f}, \overline{f}] \text{ is H-continuous} \iff \begin{cases} f \text{ is S-continuous} \\ \\ \rho(\underline{f}, \overline{f}) = 0. \end{cases}$$

Although every H-continuous function f is, in general, interval valued, the subset of the domain Ω where f assumes proper interval values is a set of first Baire category.

One of the most surprising and useful properties of the set $\mathbb{H}(\Omega)$ of all H-continuous functions is its Dedekind order completeness. What makes this property so significant is the fact that with very few exceptions the usual spaces in Real Analysis or Functional Analysis are not Dedekind order complete. The order considered in $\mathbb{H}(\Omega)$ is the one which is introduced point-wise, [1], [9].

3 The envelope viscosity solutions and Hausdorff continuous viscosity solutions

Recognizing that the concept of viscosity solution given by Definition 2 is rather weak the authors of [3] introduce the concept of envelope viscosity solution. The concept is defined in [3] for the equation (1) with Dirichlet boundary conditions. In order to keep the exposition as general as possible we will give the definition without explicitly involving the boundary condition.

Definition 6. *A function $u \in \mathcal{A}(\Omega)$ is called an envelope viscosity solution of (1) if there exist a nonempty set $\mathcal{Z}_1(u)$ of subsolutions of (1) and a nonempty set $\mathcal{Z}_2(u)$ of supersolutions of (1) such that*

$$u(x) = \sup_{f \in \mathcal{Z}_1(u)} f(x) = \inf_{f \in \mathcal{Z}_2(u)} f(x), \ x \in \Omega.$$

It is shown in [3] that every envelope viscosity solution is a viscosity solution in terms of Definition 2. Considering the concept from geometrical point of view, on can expect that by 'squeezing' the envelope viscosity solution u between a set of subsolutions and a set of supersolutions the gap between $I(u)$ and $S(u)$ would be small. However, in general this is not the case. The following example shows that the concept of envelope viscosity solution does not address the problem of the distance between $I(u)$ and $S(u)$. Hence one can have envelope viscosity solutions of little practical meaning similar to the viscosity solution in Example 1.

Example 2. Consider the following equation on $\Omega = (0,1)$

$$-u(x)(u'(x))^2 = 0, \ x \in \Omega. \tag{6}$$

For every $\alpha \in \Omega$ we define the functions

$$\phi_\alpha(x) = \begin{cases} 1 \text{ if } x = \alpha \\ 0 \text{ if } x \in \Omega \setminus \{\alpha\} \end{cases}, \quad \psi_\alpha(x) = \begin{cases} 0 \text{ if } x = \alpha \\ 1 \text{ if } x \in \Omega \setminus \{\alpha\}. \end{cases}$$

We have $\phi_\alpha \in USC(\Omega)$, $\psi_\alpha \in LSC(\Omega)$, $\alpha \in \Omega$. Furthermore, for every $\alpha \in (0,1)$ the function ϕ_α is a subsolution of (6) while ψ_α is a supersolution of (6). Indeed, both functions satisfy the equation for all $x \in \Omega \setminus \{\alpha\}$ and at $x = \alpha$ we have

$$-\phi_\alpha(\alpha)p^2 = -p^2 \leq 0 \ \text{ for all } \ p \in D^+\phi_\alpha(\alpha) = (-\infty, \infty)$$
$$-\psi_\alpha(\alpha)p^2 = 0 \geq 0 \ \text{ for all } \ p \in D^-\psi_\alpha(\alpha) = (-\infty, \infty).$$

We will show that the function

$$u(x) = \begin{cases} 1 \text{ if } x \in \Omega \setminus \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{Q} \bigcap \Omega \end{cases}$$

is an envelope viscosity solution of (6). Define $\mathcal{Z}_1(u) = \{\phi_\alpha : \alpha \in \Omega \setminus \mathbb{Q}\}$, $\mathcal{Z}_2(u) = \{\psi_\alpha : \alpha \in \mathbb{Q} \bigcap \Omega\}$. Then u satisfies $u(x) = \sup_{w \in \mathcal{Z}_1(u)} w(x) = \inf_{w \in \mathcal{Z}_2(u)} w(x)$, which implies that it is an envelope viscosity solution. Clearly neither u nor $F(u)$ is a Hausdorff continuous function. In fact we have $F(u)(x) = [0,1]$, $x \in \Omega$.

The next interesting question is whether every H-continuous solution is an envelope viscosity solution. Since the concept of envelope viscosity solutions requires the existence of sets of subsolutions and supersolutions respectively below and above an envelope viscosity solution then an H-continuous viscosity solution is not in general an envelope viscosity solution, e.g. when the H-continuous viscosity solutions does not have any other subsolutions and supersolutions around it. However in the essential case when the H-continuous viscosity solution is a supremum of subsolutions or infimum of supersolutions it can be linked to an envelope viscosity solution as stated in the next theorem.

Theorem 1. *Let $u = [\underline{u}, \overline{u}]$ be an H-continuous viscosity solution of (1) and let*

$$Z_1 = \{w \in USC(\Omega) : w\text{-subsolution}, \ w \leq \underline{u}\}$$
$$Z_2 = \{w \in LSC(\Omega) : w\text{-supersolution}, \ w \geq \overline{u}\}.$$

a) If $Z_1 \neq \emptyset$ and $\underline{u}(x) = \sup_{w \in Z_1} w(x)$ then $\underline{u}$ is an envelope viscosity solution.

b) If $Z_2 \neq \emptyset$ and $\overline{u}(x) = \inf_{w \in Z_2} w(x)$ then $\overline{u}$ is an envelope viscosity solution.

Proof. a) We choose the sets $\mathcal{Z}_1(\underline{u})$ and $\mathcal{Z}_2(\underline{u})$ required in Definition 6 as follows: $\mathcal{Z}_1(\underline{u}) = Z_1$, $\mathcal{Z}_2(\underline{u}) = \{\underline{u}\}$. Then we have $\underline{u}(x) = \sup_{w \in \mathcal{Z}_1(\underline{u})} w(x) = \inf_{w \in \mathcal{Z}_2(\underline{u})} w(x)$, which implies that $\underline{u}$ is an envelope viscosity solution. The proof of b) is done in a similar way.

Let us note that if the conditions in both a) and b) in the above theorem are satisfied then both $\underline{u}$ and $\overline{u}$ are envelope viscosity solutions and in this case in makes even more sense to consider instead the H-continuous function u.

4   Existence and uniqueness of Hausdorff continuous viscosity solutions

One of the primary virtues of the theory of viscosity solutions is that it provides very general existence and uniqueness theorems, [6]. In this section we give existence and uniqueness theorems for H-continuous viscosity solutions in a similar form to the respective theorems for continuous solutions, [6] (Theorem 4.1), and for general discontinuous solutions, [8] (Theorem 3.1), [3] (Theorem V.2.14).

Theorem 2. *(Existence) Assume that there exists Hausdorff continuous functions* $u_1 = [\underline{u}_1, \overline{u}_1]$ *and* $u_2 = [\underline{u}_2, \overline{u}_2]$ *such that* $\overline{u}_1$ *is a subsolution of (1),* $\underline{u}_2$ *is a supersolution of (1) and* $u_1 \leq u_2$*. Then there exists a Hausdorff continuous solution* u *of (1) satisfying the inequalities* $u_1 \leq u \leq u_2$*.*

The proof of the above theorem, similar to the other existence theorems in the theory of viscosity solutions, uses Perron's method and the solution is constructed as a supremum of a set of subsolutions, this time the supremum being taken in the poset $\mathbb{H}(\Omega)$ and not point-wise. We should note that due to the fact that the the poset $\mathbb{H}(\Omega)$ is Dedekind order complete it is an appropriate medium for such an application of Perron's method.

As in the theory of continuous vicosity solutions, the uniqueness result is based on the comparison principle, suitably extended for H-continuous functions.

Definition 7. *The Hamilton-Jacobi equation (1) satisfies the* comparison principle *on the set* $\mathbb{H}(\overline{\Omega})$ *of all H-continuous functions on* $\overline{\Omega}$ *if for any subsolution* $u \in \mathbb{H}(\overline{\Omega})$ *and any supersolution* $v \in \mathbb{H}(\overline{\Omega})$ *we have*

$$(u(x) \leq v(x), \ x \in \partial\Omega) \implies (u(x) \leq v(x), \ x \in \Omega).$$

Theorem 3. *(Uniqueness) Let equation (1) satisfy the comparison principle. If* $u, v \in \mathbb{H}(\overline{\Omega})$ *are Hausdorff continuous viscosity solutions such that* $u(x) = v(x), \ x \in \partial\Omega$*, then* $u(x) = v(x), \ x \in \Omega$*.*

5   Conclusion

The Hausdorff continuous functions, being a particular class of interval valued functions, belong to what is usually called Interval Analysis, see [10]. Nevertheless, recent results have shown that they can provide solutions to problems formulated in terms of point valued functions. A long outstanding problem related to the Dedekind order completion of spaces $C(X)$ of real valued continuous functions on rather arbitrary topological spaces X was solved through Hausdorff continuous functions, [1]. Following this breakthrough a significant improvement of the regularity properties of the solutions obtained through the order completion method, see [11], was reported in [2]. Namely, it was shown that these solutions can be assimilated with the class of Hausdorff continuous functions on the open domains Ω.

In this paper the Hausdorff continuous functions are linked with the concept of viscosity solutions. As shown in the Introduction the definition of viscosity solution, see Definition 2, has an implicit interval character since it places requirements only on the upper semi-continuous envelope $S(u)$ and the lower semi-continuous envelope $I(u)$. For a Hausdorff continuous viscosity solution u the functions $I(u)$ and $S(u)$ are as close as they can be in the sense of the Hausdorff distance ρ defined in [12], namely, we have $\rho(I(u), S(u)) = 0$. Hence, the requirement that a viscosity solution is Hausdorff continuous has a direct interpretation which we find clearer than the requirements related to some other concepts of discontinuous viscosity solutions. It is demonstrated that the H-continuous viscosity solutions are better then the envelope solutions for singling out physically meaningful discontinuous solutions. Furthermore, the main existence and uniqueness results can be extended to H-continuous functions in a natural way.

References

1. R. Anguelov, Dedekind order completion of C(X) by Hausdorff continuous functions, *Quaestiones Mathematicae*, **27**, 2004, 153-170.
2. R. Anguelov and E. E. Rosinger, Solving Large Classes of Nonlinear Systems of PDE's, *Computers and Mathematics with Applications* **53**, 2007, pp. 491-507..
3. M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkäuser, Boston - Basel - Berlin, 1997.
4. G. Barles, Discontinuous viscosity solutions of first order Hamilton-Jacobi Equations: A guided visit, *Nonlinear Analysis, Theory, Methods and Applications*, **20**(9), 1993, pp. 1123-1134.
5. E. N. Barron and R. Jensen, Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians, *Communications in Partial Differential Equations*, **15**(12), 1990, pp. 1713-1742.
6. M. G. Crandal, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bulletin of the American Mathematical Society*, **27**(1), 1992, pp. 1-67.
7. W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, 1993.
8. H. Ishii, Perron's method for Hamilton-Jacobi equations, *Duke Mathematical Journal*, **55**(2), 1987, pp. 369-384.
9. S. Markov, Extended interval arithmetic involving infinite intervals, *Mathematica Balkanica* **6** (1992), 269–304.
10. R. E. Moore, Methods and Applications of Interval Analysis, SIAM, Philadelphia, 1979.
11. M.B. Oberguggenberger, E.E. Rosinger, Solution on Continuous Nonlinear PDEs through Order Completion, North-Holland, Amsterdam, London, New York, Tokyo, 1994.
12. B. Sendov, Hausdorff Approximations, Kluwer, 1990.